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# IRREDUCIBLE TRIANGULATIONS OF LOW GENUS SURFACES

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Abstract. The complete sets of irreducible triangulations are known for the orientable surfaces with genus of 0, 1, or 2 and for the nonorientable surfaces with genus of 1, 2, 3, or 4. By examining these sets we determine some of the properties of these irreducible triangulations.

## 1. Introduction

The irreducible triangulations of a surface provide a basis for obtaining all the triangulations of that surface. We can sequentially contract edges of a triangulation until an irreducible triangulation is produced. Reversing this sequence we can produce any triangulation of a surface with a sequence of vertex splittings starting with an irreducible triangulation. Thus all the triangulations of a surface can be generated from the irreducible triangulations of that surface by vertex splittings. The irreducible triangulations of a surface can be used to actually generate [24] the triangulations of the surface.

Irreducible triangulations can also be used to check properties which are preserved by vertex splitting. For example, let $\mathcal{P}$ be a property possessed by some of the triangulations of the surface $S$ which is preserved by vertex splitting such as "contains a cycle which separates the surface $S$". If every irreducible triangulations of $S$ possesses $\mathcal{P}$ then every triangulations of $S$ possesses $\mathcal{P}$. Conversely, if there is a counterexample to "all triangulations of $S$ possess $\mathcal{P}$" then there is a counterexample among the irreducible triangulation of $S$.

For any fixed surface the number of irreducible triangulations is finite [1]. Irreducible triangulations have been determined and displayed by a number of authors: the single irreducible triangulation of the sphere ($S_0$) by Steinitz and Rademacher [19]; the two irreducible triangulations of the projective plane or the cross surface ($N_1$) by Barnette [2]; the 21 irreducible triangulations of the torus ($S_1$) by Lawrencenko [9]); and the 29 irreducible triangulations of the Klein bottle ($N_2$) by Lawrencenko and Negami [10] and Sulanke [23]. The irreducible triangulations of the double torus ($S_2$), the triple cross surface ($N_3$), and the quadruple cross surface ($N_4$) have been generated by the author using a computer program [22].

## 2. Definitions

A *triangulation* of a closed surface is a simple graph embedded in the surface such that each face is a triangle and any two faces share at most one edge.

In a triangulation $T$ let $abc$ and $acd$ be two faces which have $ac$ as a common edge. The *contraction* of $ac$ is obtained by deleting $ac$, identifying vertices $a$ and $c$, removing one of the multiple edges $ab$ or $cb$, and removing one of the multiple

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edges $ad$ or $cd$. The edge $ac$ of a triangulation $T$ is *contractible* if the contraction of $ac$ yields another triangulation of the surface in which $T$ is embedded. If the edge $ac$ is contained in a 3-cycle other than the two which bound the faces which share it then its contraction would produce multiple edges. Thus, for a triangulation $T$, not $K_4$ embedded in the sphere, an edge of $T$ is not contractible if and only if that edge is contained in at least three 3-cycles. A triangulation is said to be *irreducible* if it has no contractible edges.

The operation of *splitting* a vertex is the reverse of contracting an edge. In a triangulation let $ab$ and $ac$ be two distinct edges. The *splitting* of the vertex $a$ (along the edges $ab$ and $ac$) is obtained by creating a new vertex $a'$, three new edges $a'a$, $a'b$, and $a'c$, and two new faces $a'ab$ and $a'ac$. The triangulation obtained by splitting a vertex is embedded in the same surface as the original triangulation.

We denote the orientable surface with genus $g$, the sphere with $g$ handles attached, as $S_g$ and the nonorientable surface with genus $g$, the sphere with $g$ crosscaps attached, as $N_g$. Define the *Euler genus* of the surface $S$ to be $2 - \chi(S)$. For orientable surfaces the Euler genus is twice the genus and for nonorientable surfaces the Euler genus is the same as the genus.

## 3. Generating irreducible triangulations

The author has recently developed an algorithm [22] for generating irreducible triangulations of a surface by using the irreducible triangulations of other surfaces with smaller Euler genera. This algorithm was implemented as a computer program. The irreducible triangulations of $S_2$, $N_3$, and $N_4$ were generated and are available as computer files [24].

Before we briefly describe the algorithm used to generate irreducible triangulations we examine how an irreducible triangulation can be reduced to an irreducible triangulation with a lower genus. For simplicity we only consider orientable surfaces here. Let $T$ be an irreducible triangulation of $S_g$ with $g > 0$. Every edge of $T$ is on a 3-cycle which is not a face. Many of these 3-cycles do not separate $S_g$ into two components. Pick one of these nonseparating 3-cycles. Cut $T$ along this 3-cycle thereby cutting one of the handles of $S_g$. Cap the resulting two holes with new triangular faces to produce a new triangulation $T'$ of $S_{g-1}$. Contract contractible edges until an irreducible triangulation of $S_{g-1}$ is obtained.

To generate an irreducible triangulation of $S_g$ we reverse these steps in effect "growing a handle". Start with an irreducible triangulation of $S_{g-1}$. Split vertices checking each new triangulation to see if it can be used to form an irreducible triangulation of $S_g$. The final step is the reverse of the cut and cap described above. Remove two faces and join the resulting boundary cycles in such a way that the resulting triangulation is still orientable.

An irreducible triangulation of $N_g$ can be generated in a similar way by "growing a handle or a crosshandle". Start with an irreducible triangulation of $S_{g/2-1}$ or $N_{g-2}$ and split vertices. In the final step we remove two faces and join the resulting boundary cycles in such a way that the resulting triangulation is nonorientable.

We can also "grow a crosscap" to generate an irreducible triangulation of $N_g$ starting with an irreducible triangulation of $S_{(g-1)/2}$ or $N_{g-1}$. As new triangulations are produced by edge splitting we check for vertices with degree 6. When we remove a vertex with degree 6 and its incident faces a hole with a 6-cycle as a

| Vertices | $S_1$ | $S_2$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
|---|---|---|---|---|---|---|
| 6 | | | 1 | | | |
| 7 | 1 | | 1 | | | |
| 8 | 4 | | | 6 | | |
| 9 | 15 | | | 19 | 133 | 37 |
| 10 | 1 | 865 | | 2 | 2521 | 10347 |
| 11 | | 26276 | | 2 | 4638 | 370170 |
| 12 | | 117047 | | | 1320 | 1891557 |
| 13 | | 159205 | | | 946 | 2067817 |
| 14 | | 54527 | | | 93 | 956967 |
| 15 | | 38195 | | | 50 | 700733 |
| 16 | | 664 | | | 7 | 186999 |
| 17 | | 5 | | | | 89036 |
| 18 | | | | | | 19427 |
| 19 | | | | | | 3975 |
| 20 | | | | | | 832 |
| 21 | | | | | | 79 |
| 22 | | | | | | 6 |
| Total | 21 | 396784 | 2 | 29 | 9708 | 6297982 |

TABLE 1. Irreducible triangulation by vertices

boundary is produced. By identifying the 3 pairs of opposite vertices on this 6-cycle we check if the result is an irreducible triangulation.

## 4. COUNTS

Due to the large number of irreducible triangulations of $S_2$, $N_3$, and $N_4$ the irreducible triangulations are not be displayed here but some of their properties are presented. For comparison we also include similar properties for $S_0$, $S_1$, $N_1$, and $N_2$.

Table 1 shows for each surface the number of irreducible triangulations having a given number of vertices.

## 5. NONCONTRACTIBLE SEPARATING CYCLES

Let $v_1 v_2 \ldots v_n$ be an n-cycle in a graph embedded on the surface $S$ and let $C$ be the closed curve which is the embedding of $v_1 v_2 \ldots v_n$ in $S$. $v_1 v_2 \ldots v_n$ is *separating* if $S - C$ is disconnected. $v_1 v_2 \ldots v_n$ is *contractible* if a component of $S - C$ is a 2-cell, otherwise, it is *noncontractible*. This definition of a contractible cycle should not be confused with the definition of a contractible edge given earlier. Necessary conditions for the existence of a *noncontractible separating cycle* or *NSC* have been studied [5] [18] [26]. An NSC separates a surface into two components neither of which is a 2-cell. Thus a surface having an NSC must have genus greater than 1.

The existence of an NSC in a triangulation and the genera of the separated surfaces are preserved by vertex splitting. Thus if every irreducible triangulation of a surface has an NSC then every triangulation of that surface has an NSC.

Barnette conjectured that every triangulation of a surface with genus greater than 1 has an NSC. Lawrencenko and Negami [10] showed that every irreducible

triangulation of $N_2$ has an NSC and thus every triangulation of $N_2$ has an NSC. Ellingham, Zha, and Jennings [6] have shown (without any reference to irreducible triangulations) that every triangulation of $S_2$ has an NSC.

By checking that each irreducible triangulation of $S_2$, $N_2$, $N_3$, and $N_4$ has an NSC we have the following result which is new only for $N_3$, and $N_4$.

**Theorem 1.** *Every triangulation of $S_2$, $N_2$, $N_3$, or $N_4$ has an NSC.*

Similarly, if an NSC separates a surface with Euler genus $g$ into two surfaces with Euler genera $h$ and $g-h$ then any triangulation obtained by vertex splitting of this triangulation has an NSC which separates the surface into two surfaces with Euler genera $h$ and $g-h$.

Thomassen conjectured ([11] page 167) that given a triangulation of an orientable surface with genus $g$ and an integer $h$ such that $1 \leq h < g$, then the triangulation must contain an NSC such that the two surfaces separated by the NSC (after capping the holes with disks) have genera $h$ and $g-h$, respectively. This conjecture is equivalent to Barnette's conjecture for $S_2$ (and $S_3$) but we can make a similar conjecture for nonorientable surfaces.

**Conjecture 1.** *Given a triangulation of a nonorientable surface with Euler genus $g$ and an integer $h$ such that $1 \leq h < g$, then the triangulation must contain an NSC such that the two surfaces separated by the NSC have Euler genera $h$ and $g-h$, respectively.*

By checking the irreducible triangulations of $N_4$ we have the following.

**Theorem 2.** *Every triangulation of $N_4$ has an NSC which separates the surface into two surfaces each with Euler genus 2. Every triangulation of $N_4$ has an NSC which separates the surface into two surfaces with Euler genera 1 and 3, respectively.*

From Theorems 1 and 2 it follows that Conjecture 1 is true for $N_2$, $N_3$, and $N_4$. Conjecture 1 and Theorem 2 do not specify the orientability of the separated surfaces. For example, there are triangulations of $N_3$ which do not have an NSC which separates the surface into $N_1$ and $N_2$. Such an example can be constructed using any irreducible triangulation of $N_1$ and any irreducible triangulation of $S_1$. Remove a face from each of these two irreducible triangulations and identify the resulting boundaries. Let $C_1$ be the closed curve in $N_3$ where the two surfaces were joined. Assume there is an NSC which separates $N_3$ into $N_1$ and $N_2$. Let $C_2$ be the closed curve in $N_3$ corresponding to this NSC. Due to the topology of $N_3$ the curves $C_1$ and $C_2$ must cross at least four times. But this contradicts the fact that the cycle corresponding to $C_1$ has length 3. Similarly, there are also triangulations of $N_3$ which do not have an NSC which separates the surface into $N_1$ and $S_1$.

For $N_3$, there are 9184 irreducible triangulations which have an NSC which separates the surface into $N_1$ and $N_2$ and there are 8533 irreducible triangulations which have an NSC which separates the surface into $N_1$ and $S_1$.

For $N_4$, there are 6062415 irreducible triangulations which have an NSC which separates the surface into $N_2$ and $N_2$ and there are 5971981 irreducible triangulations which have an NSC which separates the surface into $N_2$ and $S_1$.

The *edge-width* of a triangulation is the length of the shortest NSC in the triangulation. Tables 2 through 5 show the number of irreducible triangulations for a given number of vertices and a given value of the edge-width.

| | Edge-width | | | | | |
|---|---|---|---|---|---|---|
| Vertices | 3 | 4 | 5 | 6 | 7 | 8 |
| 10 | | 2 | 51 | 681 | 130 | 1 |
| 11 | 2 | 58 | 2249 | 16138 | 7818 | 11 |
| 12 | 25 | 1516 | 20507 | 72001 | 22877 | 121 |
| 13 | 710 | 13004 | 50814 | 78059 | 16609 | 9 |
| 14 | 8130 | 30555 | 12308 | 3328 | 205 | 1 |
| 15 | 36794 | 1395 | 3 | 1 | 2 | |
| 16 | 661 | 3 | | | | |
| 17 | 5 | | | | | |

Table 2. Irreducible triangulation of $S_2$ by vertices and edge-width

| | Edge-width | | | |
|---|---|---|---|---|
| Vertices | 3 | 4 | 5 | 6 |
| 8 | | 1 | 5 | |
| 9 | 1 | 5 | 2 | 11 |
| 10 | 1 | 1 | | |
| 11 | 2 | | | |

Table 3. Irreducible triangulation of $N_2$ by vertices and edge-width

| | Edge-width | | | |
|---|---|---|---|---|
| Vertices | 3 | 4 | 5 | 6 |
| 9 | | 1 | 119 | 13 |
| 10 | 1 | 140 | 1862 | 518 |
| 11 | 72 | 1248 | 1558 | 1760 |
| 12 | 502 | 811 | 4 | 3 |
| 13 | 912 | 34 | | |
| 14 | 93 | | | |
| 15 | 50 | | | |
| 16 | 7 | | | |

Table 4. Irreducible triangulation of $N_3$ by vertices and edge-width

## 6. Nonseparating cycles

Every cycle of a triangulation of $S_0$ separates and we exclude such triangulations in this section. In [22] it is shown that for every vertex of an irreducible triangulation there are at least two nonseparating 3-cycles containing that vertex. Thus every irreducible triangulation has a nonseparating cycle and, therefore, every triangulation has a nonseparating cycle. For triangulations of orientable surfaces the only topological type of a nonseparating cycle is one which cuts a handle.

Let $S$ be a triangulated nonorientable surface with a nonseparating cycle. Let $C$ be the closed curve which the embedding of that cycle in $S$. The cycle is *one-sided* if the neighborhood of $C$ in $S$ is homeomorphic to a Möbius band, otherwise the cycle is *two-sided*. The cycle is *orientable-leaving* if $S - C$ is orientable, otherwise the

| | Edge-width | | | | | |
|---|---|---|---|---|---|---|
| Vertices | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | | | 17 | 20 | | |
| 10 | | | 5028 | 5222 | 97 | |
| 11 | | 4503 | 209623 | 150994 | 5050 | |
| 12 | 2499 | 161502 | 983249 | 717138 | 27169 | |
| 13 | 76309 | 704856 | 698076 | 566851 | 21723 | 2 |
| 14 | 396148 | 519038 | 36649 | 5066 | 66 | |
| 15 | 633195 | 67538 | | | | |
| 16 | 181884 | 5115 | | | | |
| 17 | 88799 | 237 | | | | |
| 18 | 19427 | | | | | |
| 19 | 3975 | | | | | |
| 20 | 832 | | | | | |
| 21 | 79 | | | | | |
| 22 | 6 | | | | | |

TABLE 5. Irreducible triangulation of $N_4$ by vertices and edge-width

cycle is *nonorientable-leaving*. For triangulations of nonorientable surfaces there are four possible topological types of nonseparating cycles depending on whether it is one- or two-sided and whether it is orientable- or nonorientable-leaving. At most three of the these types of nonseparating cycles can occur for a fixed nonorientable surface since orientable surfaces have an even Euler genus.

By checking the irreducible triangulations of $N_1$, $N_2$, $N_3$, and $N_4$ we have the following theorem.

**Theorem 3.** *Every triangulation of $N_1$ has a nonseparating cycle which is one-sided and orientable-leaving.*

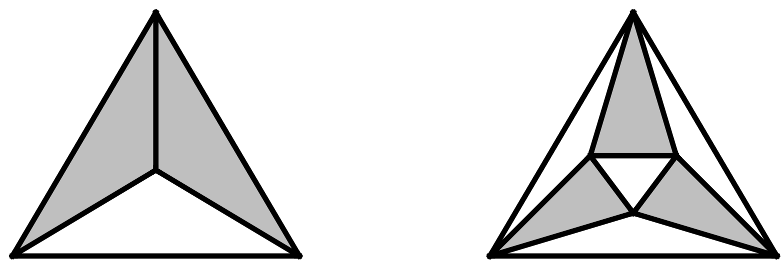
*Every triangulation of $N_2$ has a nonseparating cycle which is one-sided and nonorientable-leaving and a nonseparating cycle which is two-sided and orientable-leaving.*

*Every triangulation of $N_3$ has a nonseparating cycle which is one-sided and orientable-leaving; a nonseparating cycle which is one-sided and nonorientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.*

*Every triangulation of $N_4$ has a nonseparating cycle which is one-sided and nonorientable-leaving; a nonseparating cycle which is two-sided and orientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.*

**Conjecture 2.** *If $g >= 3$ is odd then every triangulation of $N_g$ has a nonseparating cycle which is one-sided and orientable-leaving; a nonseparating cycle which is one-sided and nonorientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.*

*If $g >= 4$ is even then every triangulation of $N_g$ has a nonseparating cycle which is one-sided and nonorientable-leaving; a nonseparating cycle which is two-sided and orientable-leaving; and a nonseparating cycle which is two-sided and nonorientable-leaving.*



$B_3$ $B_4$

FIGURE 1. Base triangulations for constructing large irreducible triangulations

## 7. Maximal irreducible triangulations

Define $V_{max}(S)$ to be the maximum number of vertices in an irreducible triangulation of $S$. From Tables 2 through 5 we see that if an irreducible triangulation $T$ of one of these surfaces $S$ has $|V(T)| = V_{max}(S)$ then $T$ has an NSC of length 3. For $S_2$ this confirms a conjecture of Negami [13]. This suggests that maximal irreducible triangulations are made up of other triangulations joined at a single face of each.

We can use the following construction to obtain large irreducible triangulations which give a lower bound for $V_{max}(S)$. This construction is similar to the one given by Nakamoto and Ota [12] and the lower bound which it provides is a slight improvement.

For $N_1$ there is only one maximal irreducible triangulation $M$ which has 7 vertices. If we take two copies of $M$ and remove one face from each we can join them at the boundaries of these faces to obtain a triangulation of $N_2$. This triangulation has 11 vertices and is irreducible.

For each $g > 2$ we will construct a base triangulation $B_g$ of $S_0$ which when joined with $g$ copies of $M$ will produce an irreducible triangulation of $N_g$.

The left side of Figure 1 shows $B_3$ which is a triangulation of $S_0$ from which three faces (the shaded faces and the outside face) have been removed. Every edge of $B_3$ is on a removed face. If we join three punctured copies of $M$ at these faces we get a triangulation of $N_3$. This triangulation is irreducible since each edge of $B_3$ is now on at least three 3-cycles.

The right side of Figure 1 shows $B_4$ which is a triangulation of $S_0$ from which four faces have been removed. Again every edge of $B_4$ is on a removed face. When we join four punctured copies of $M$ at the removed faces we obtain an irreducible triangulation of $N_4$.

If we take two copies of $B_4$ and join them at removed faces then we obtain $B_6$ which is a triangulation of $S_0$ with 6 faces removed. Every edge of $B_6$ is either on a removed face or on at least three 3-cycles. Joining six punctured copies of $M$ we obtain an irreducible triangulation of $N_6$. We can repeat this construction to obtain $B_g$ for even $g > 2$. $|V(B_4)| = 6$ and each additional copy of $B_4$ adds 3 vertices such that $|V(B_g)| = 3g/2$. Each copy of $M$ adds 4 vertices. Thus for even $g > 2$ the number of vertices in the constructed irreducible triangulation of $N_g$ is $11g/2$.

To obtain $B_g$ for odd $g$ we join $B_3$ to $B_{g-1}$. Then $|V(B_g)| = 3(g-1)/2 + 1 = 3g/2 - 1/2$ and the number of vertices in the constructed irreducible triangulation of $N_g$ is $11g/2 - 1/2$.

Thus for any $g$ we have

$$V_{max}(N_g) \geq \lfloor 11g/2 \rfloor$$

For $S_1$ the only maximal irreducible triangulation has 10 vertices. Repeating the above construction with this triangulation as $M$ we obtain

$$V_{max}(S_g) \geq \lfloor 17g/2 \rfloor$$

In the above construction the triangulation $M$ does not need to be irreducible. Any edge of the removed face may be contractible and the resulting triangulation would still be irreducible.

A triangulation is *almost irreducible* if it is not irreducible and it has a face which is incident on all the contractible edges. If $M$ is almost irreducible then the construction still produces an irreducible triangulation. However, there are no almost irreducible triangulations of $N_1$ [10] and there are no almost irreducible triangulations $T$ of $S_1$ for which $|V(T)| > V_{max}(S_1)$. There are 8 almost irreducible triangulations of $S_1$ but none have more than 9 vertices [20].

## 8. Pseudo-minimal triangulations

Two triangulations $T$ and $T'$ of a surface are *equivalent* if there is a isomorphism $h$ with $h(T) = T'$. That is, if $a$, $b$, and $c$ are vertices of $T$ then $ab$ is an edge of $T$ if and only if $h(a)h(b)$ is an edge of $T'$ and a face of $T$ is bounded by the cycle $abc$ if and only if a face of $T'$ is bounded by the cycle $h(a)h(b)h(c)$.

Let $ac$ be an edge in a triangulation $T$ and $abc$ and $acd$ be the two faces which have $ac$ as a common edge. The *diagonal flip* of $ac$ is obtained by deleting $ac$, adding edge $bd$, deleting the faces $abc$ and $acd$, and adding the faces $abd$ and $bcd$. An edge $ac$ of a triangulation $T$ is *flippable* if the diagonal flip of $ac$ yields another triangulation of the surface in which $T$ is embedded. Thus the edge $ac$ is flippable if there is not already an edge $bd$. Two triangulations are *equivalent under diagonal flips* if one is equivalent to a triangulation obtained from the other by a sequence of diagonal flips.

The number of vertices of an irreducible triangulation can not be reduced by edge contraction. Negami [14] defines a type of triangulation for which the number of vertices can not be reduced by a combination of diagonal flips and edge contractions. An irreducible triangulation is said to be *pseudo-minimal* if it is equivalent under diagonal flips only to irreducible triangulations.

A triangulation is said to be *minimal* if there are no triangulations of the same surface with fewer vertices. It is clear that such a triangulation is also pseudo-minimal. The number of vertices in a minimal triangulation for nonorientable surfaces was determined by Ringel [17] and for orientable surfaces by Jungerman and Ringel [7]. It is given for all surface except $N_2$, $N_3$, and $S_2$ by the formula:

$$V_{min}(S) = \left\lceil \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rceil$$

For the three exceptions the value is one more than the value given by the formula: $V_{min}(N_2) = 8$, $V_{min}(N_3) = 9$, and $V_{min}(S_2) = 10$.

Let $N(S)$ be the minimum value such that two triangulations $T$ and $T'$ of $S$ are equivalent under diagonal flips if $|V(T)| = |V(T')| \geq N(S)$. Negami [14] has shown that such a finite value exists for any $S$.

$N(S_0) = V_{min}(S_0) = 4$, $N(S_1) = V_{min}(S_1) = 7$, $N(N_1) = V_{min}(N_1) = 6$, and $N(N_2) = V_{min}(N_2) = 8$ are known [25] [4] [16].

Checking the irreducible triangulations generated for $S_2$ we have determined that the 865 minimal triangulations are the only pseudo-minimal triangulations and that these pseudo-minimal triangulations form one equivalence class under diagonal flips. Thus $N(S_2) = V_{min}(S_2) = 10$ ([14] [21]). Similarly, the 133 minimal triangulations $N_3$ are the only pseudo-minimal triangulations and they form one equivalence class under diagonal flips. Thus $N(N_3) = V_{min}(N_3) = 9$.

The situation for $N_4$ is more interesting. The 37 minimal triangulations are the only pseudo-minimal triangulations. However, these pseudo-minimal triangulations are partitioned into three equivalence classes under diagonal flips [21] with cardinality 32, 3, and 2. Using this complete list of pseudo-minimal triangulations of $N_4$ it is possible to show [21] that $N(N_4) = V_{min}(N_4) + 1 = 10$.

Suppose for a surface $S$ there exist at least two inequivalent minimal triangulations which have no flippable edges. Then $N(S) > V_{min}(S)$. There are an infinite number of surfaces which have many inequivalent triangular embeddings of complete graphs [15] [8] [3]. A triangular embeddings of complete graph is minimal and a complete graph has no flippable edges. Therefore there are an infinite number of surfaces $S$ for which $N(S) > V_{min}(S)$. For each of the surfaces $S_g, 3 <= g <= 15$ and $N_g, 5 <= g <= 30$, the author has found, using random computer searching [21], a pair of minimal triangulations which are inequivalent under diagonal flips. The existence of these pairs shows that if $3 <= g <= 15$ then $N(S_g) > V_{min}(S_g)$ and that if $4 <= g <= 30$ then $N(N_g) > V_{min}(N_g)$.

**Conjecture 3.** *The only surfaces $S$ for which $N(S) = V_{min}(S)$ are $S_0$, $S_1$, $S_2$, $N_1$, $N_2$, and $N_3$.*

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